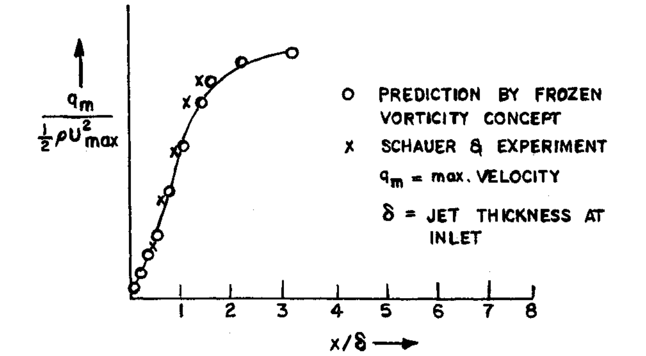

Fig. 4 Prediction of static pressure distribution by frozen vorticity concept.

$$\left| \frac{\psi_p^k - \psi_p^{k-1}}{\psi_{max}} \right| < \epsilon$$

where ϵ is a small number. ϵ in the present study was chosen to be 10^{-4}.

Boundary Conditions

The inlet boundary conditions on $A'B'$ are fixed by upstream velocity profile given by Eq. (1). The wall boundary conditions are fixed by $\psi = 0$. At the outlet $A'x'$ and $B'y'$ it was assumed that boundary layer conditions were met again gives a profile given by Eq. (1).

Results

Figures 3 and 4 illustrate the plot of the velocity maxima predicted by the above method compared with the measurements of Schauer and Eustis.[6] The nondimensional static pressure on the bottom wall is also calculated by using Bernoulli's theorem.

$$p + 1/2\rho u^2 = \text{const}$$

on a streamline. The agreement with the experiments of Refs. 3 and 6 is good for the measured static pressure distribution. It is also seen from the above graphs that a nondimensional profile could be obtained for various nozzle distances. The computing time involved in the above calculations were of the order of 5 secs. The boundary layer close to the wall could be described by a Boundary layer type approximation with a velocity at the outer edge given by the velocity distribution on the bottom plate.

Conclusions

Considering the normal impinging jet flow as consisting of two different regimes an expeditive method of calculation has been developed. The present method takes into account the viscosity effects producing a rotation in the flow, and also it requires an incomparably shorter computer time than needed for the solution of the full Navier-Stokes equation. The accuracy of the predictions by using this method is as good as the predictions assumed by more time consuming methods.

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On Calculation of Induced Drag and Conditions Downstream of a Lifting Wing

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THIS Note concerns the calculation of the induced drag of a wing in terms of conditions far behind it—in the "Trefftz Plane." As every student of wing theory is taught, the lift can easily be calculated by momentum principles: it is equal to the rate of increase of downwash momentum (more precisely, *impulse*) and therefore to the downward impulse in a unit slab of two-dimensional flow downstream. But when the drag is sought it is customary to resort to an energy argument, and it is easily ascertained that the drag of any wing in steady flight is equal to the kinetic energy in the same unit slab downstream.[1,2] It is instructive, however, to carry out the drag calculation by momentum principles; the calculation has certain subtleties and casts light on some interesting facts concerning the flow downstream of a lifting wing-system—a topic of considerable current interest.

The subtleties mentioned arise from the fact that, for a lightly loaded wing (small-perturbation flow), the drag is a second-order quantity. It is necessary to account for the first-order deflection of the wake; hence, the configuration under consideration is that sketched in Fig. 1, which shows the wing and trailing-vortex wake in a wing-fixed frame of reference (steady flow).

At this point we fix our attention specifically on the wing of elliptic lift distribution, for which the downwash at the vortex sheet far downstream is uniform across the span and equal to $(2C_\beta/\pi R)V$ or $4_\beta/\pi\rho Vn^2$, where C_β denotes the lift coefficient, R the aspect ratio b^2/S, L the lift, ρ the fluid density, b the wing span, and S the wing area. To emphasize a peculiarity of this situation, however, we shall replace the factor 2 here by a symbol k,

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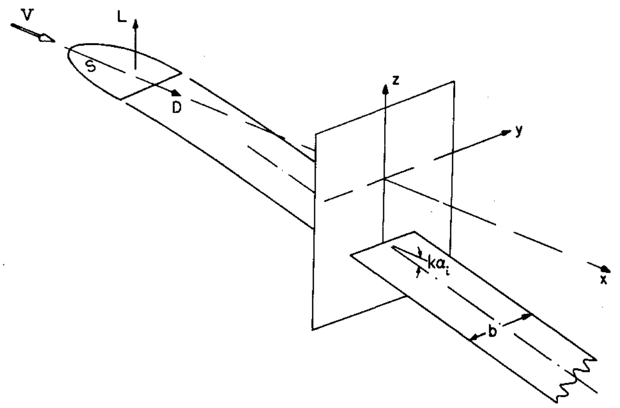

Fig. 1. Sketch showing wing, trailing-vortex wake, and "Trefftz Plane."

presumed to be unknown; viz., we shall write for the downwash (uniform across the span) far downstream

$$\text{Downwash far downstream} = k\frac{C_L}{\pi R} = 2k\frac{L}{\pi\rho V b^2}$$
$$= k\alpha_i, \text{ say} \quad (1)$$

The x-forces acting on the system are 1) the thrust, equal and opposite to the drag D, and 2) the resultant of the pressure perturbations acting on a transverse yz-plane downstream. The change of x-impulse effected by these forces is also calculated by integration over this transverse plane. The momentum balance therefore reads

$$-D - \iint(p - p_\infty)dydz = \rho V\iint(u - V)dydz \quad (2)$$

Now, the pressure is given by Bernoulli's Equation

$$p - p_\infty = \frac{\rho}{2}\{V^2 - (u^2 + v^2 + w^2)\} \quad (3)$$

where u, v, w are the usual Cartesian velocity components and the flow field far downstream consists of two-dimensional flow in planes normal to the trailing-vortex sheet, which is inclined downward at the angle $k\alpha_i$ mentioned above. Thus, u, v, and w can be expressed in terms of the velocity potential $\varphi(y, z + xk\alpha_i)$, say, of this two-dimensional flow. To second order, they are

$$u = V + \varphi_z k\alpha_i, \quad v = \varphi_y, \quad w = \varphi_z \quad (4)$$

so that Eq. (3) becomes, to second order

$$p - p_\infty = -\frac{\rho}{2}\{2V\varphi_z k\alpha_i + \varphi_y{}^2 + \varphi_z{}^2\} \quad (5)$$

and Eq. (2) becomes

$$-D + \rho V k\alpha_i\iint\varphi_z dydz + \frac{\rho}{2}\iint(\varphi_y{}^2 + \varphi_z{}^2)dydz$$
$$= \rho V k\alpha_i\iint\varphi_z dydz \quad (6)$$

or

$$D = \frac{\rho}{2}\iint(\varphi_y{}^2 + \varphi_z{}^2)dydz \quad (7)$$

But this is exactly the statement relating drag to kinetic energy, mentioned above; we have therefore arrived at the familiar result for elliptic lift distribution[1,2]

$$D = (C_L/\pi R)L = (2L^2/\pi\rho V^2 b^2) \quad (8)$$

The first interesting feature of this calculation is that the induced angle $k\alpha_i$ far downstream has disappeared from consideration. The x-impulse term on the right-hand side is $k\alpha_i$ times the z-impulse, which is $-L$; thus it is $-k$ times the drag; nevertheless it is cancelled by a pressure contribution in the left-hand side, and the correct answer is obtained for any value of k.

A second interesting feature is that the net pressure contribution is a force directed *upstream*. This follows from what has just been said, for the first integral on the left-hand side of Eq. (6) is $-kD$, namely $-2D$, and the second integral is $+D$. Thus one's usual concept—at least the author's—wherein the drag is balanced by reduced pressures downstream, seems to be incorrect. The correct picture involves generally *increased* pressures downstream, accompanied by upwind-directed induced velocities.

It is also instructive (and makes a fine homework problem!) to carry out this calculation in a stream-fixed frame of reference. It must then be remembered that the trailing-vortex wake is moving downward in this frame, and unsteady-flow formulas must be used to obtain the correct pressure. When this is done, the results are, of course, identical to those of the present note.

Finally, the generalization to a wing-system having arbitrary spanwise lift distribution can easily be carried out. As long as u is given by a second-order perturbation added to V, as is true far downstream for all small-perturbation cases, the x-impulse integral on the right-hand side of Eq. (2) is cancelled by a term in the pressure integral, and Eq. (7) results. This calculation is made by Landau and Lifshitz,[3] but their argument has serious flaws, such as the claim that the x-impulse integral

$$\rho\iint(u - V)dydz$$

is zero. As we have seen, it is actually proportional to the lift L in the classical case of elliptic lift distribution. The error arises from insufficient care in applying the principles of continuity and momentum in an infinite domain.

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Errata

On the Dependence of Materials Erosion on Environmental Parameters at Supersonic Velocities

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THE following footnote was inadvertently omitted: "Received September 28, 1972; revision received August 9, 1973."

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